Problem 1. Linear Algebra

a) What is an eigenvalue? Write the eigenvalue equation.

b) The operator \( C = AB - BA \) is anti-hermitian if both \( A \) and \( B \) are hermitian. Given the properties of \( C \) what is the definition of an anti-hermitian operator?

c) The operator \( \sigma_z \) corresponds to the component of the spin of an electron along the z-axis, in units of \( \hbar/2 \). For a state represented by the vector \( |\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \), where \( \alpha \) and \( \beta \) are complex numbers, calculate: (i) the expectation value of \( \sigma_z \); (ii) the probability that the spin z-component is positive.

d) Is the state \( |\psi\rangle = \frac{1}{2}(|a\rangle|b\rangle + |b\rangle|a\rangle + |a\rangle|a\rangle + |b\rangle|b\rangle) \) entangled?

e) Consider a flux of neutrons entering the interferometer has a random polarization. That is, each neutron has a probability \( p_\uparrow \) to have its spin pointing up and a probability \( p_\downarrow \) to be pointing down. How can you describe the state of the neutron?

Problem 2. Trace as an inner product for the matrix vector space

a) If \( A \) and \( B \) are matrices of the same dimensions, show that \( \langle A, B \rangle = Tr[A^\dagger B] \) has all the properties of an inner product.

Problem 3. Neutrino Oscillations

Neutrinos can oscillate between different flavors because of the mixing of their mass eigenstate. Observing these oscillation is the goal of experiments like Opera, that measures neutrinos produced at CERN. At CERN, neutrinos are produced from kaon/pion decay, which generates muon neutrinos, e.g. \( \pi^+ \rightarrow \mu^+ + \nu_\mu \). Muon neutrinos \( \nu_\mu \) can oscillate into tau neutrinos, \( \nu_\tau \).

a) What is the Hamiltonian \( H \) describing the interaction that makes the neutrino oscillate between the muon and tau flavor? (assume that the energy needed for this change is \( \Delta \) or equivalently, setting \( \hbar = 1 \), that the rate of the oscillation is \( \Delta \))

b) Find the normalized energy eigenkets of the Hamiltonian. What are the corresponding eigenvalues?

c) Suppose a neutrino is created in the state \( |\alpha\rangle = \alpha|\mu\rangle + \beta|\tau\rangle \) at \( t = 0 \). Find the state vector at \( t > 0 \) by applying the appropriate time-evolution operator to \( |\alpha(0)\rangle \).

d) Suppose at \( t = 0 \) a muon neutrino is produced in a reaction. What is the probability for observing tau neutrino as a function of time?

e) Suppose I suggested you that the (wrong) answer to question (a) is \( H = \Delta|\tau\rangle\langle\mu| \). By explicitly solving the time-evolution problem with this Hamiltonian, show that probability conservation is violated.

f) Now assume that different neutrinos have different energies \( \omega_\mu > \omega_\tau \). Write the Hamiltonian describing this energy difference in the \( |\mu\rangle, |\tau\rangle \) basis.

g) Calculate again the evolution of the muon neutrino (as in question (c)) when to the Hamiltonian you found in question (a) you add the Hamiltonian of question (f).
**Problem 4. Stern-Gerlach Measurement**

A beam of spin 1/2 atoms goes through a series of Stern-Gerlach-type measurements as follows:

1. The first measurement accepts $s_z = \hbar/2$ atoms and rejects $s_z = -\hbar/2$ atoms (where $s_z$ is the eigenvalue of the spin angular momentum $S_z$.)

2. The second measurement accepts $s_n = \hbar/2$ atoms and rejects $s_n = -\hbar/2$ atoms, where $s_n$ is the eigenvalue of the operator $S \cdot \hat{n}$, with $\hat{n}$ making an angle $\beta$ in the $xz$-plane with respect to the $z$-axis.

3. The third measurement accepts $s_z = -\hbar/2$ atoms and rejects $s_z = \hbar/2$ atoms.

a) What is the intensity of the final beam when the $s_z = \hbar/2$ beam surviving the first measurement is normalized to unity?

b) How must we orient the second measuring apparatus to maximize the intensity of the final beam?

**Problem 5. Neutron Interferometry**

Consider a neutron interferometer (NI), such as the Mach-Zehnder interferometer in the figure. We send in a beam of neutrons. We assume that the flux of neutrons is so low (neutrons can be very slow) so that only one neutron is present at any time inside the interferometer. The first beamsplitter divides the neutron flux into two parts, that will go into the upper arm or the lower arm. The lower and upper beams are then reflected at the mirrors and recombined at the second beam splitter, after which the neutron flux is measured at one arm. We assume that both beamsplitter works in the same way, delivering an equal flux to each arm (that is, the transmission and reflection are the same).

a) Define the (minimal) Hilbert space describing this problem (e.g. give the basis spanning the space)

b) What is the propagator describing the action of the Beamsplitter?

c) What is the state at the position 1, assuming the neutron was initially traveling upward before entering the NI?

d) What is the probability of observing a neutron in the upper arm detector?

e) We now introduce an object in the lower path. This modifies the momentum of the neutron, and its effect is seen as an added phase to the neutrons passing through the lower path. Write the operator describing this phase shift and calculate again the probability of measuring a neutron at the upper path detector.

f) The usual signal for interferometers is the contrast $C = \left| \frac{\langle S_U \rangle - \langle S_L \rangle}{\langle S_U \rangle + \langle S_L \rangle} \right|$, where $\langle S_U \rangle$ is the signal (number of neutrons) at the upper(lower) detector. Indeed, this is always necessary since we need to calibrate and normalized the signal. What is the contrast for the neutron interferometer if the added phase (see previous question) is $\phi = \pi/2$?

Bonus: what is the operator describing the observable measured by the contrast?

**Problem 6. Pure vs. Mixed States**

Consider again the NI of Problem 5. We previously assumed that the beam of neutrons all had the same (upward) momentum $|\psi(0)\rangle = |U\rangle$. However, if the neutrons arrive from a reactor, we might not be able to control their initial state.
b) Assume the neutrons can be represented by a mixed-state, with 50-50 probability of having initially an upward \( |U \rangle \) or downward \( |L \rangle \) momentum. What is the operator describing this state?

c) For the initial state described above, what is the measured contrast for the same condition as in Problem 5.? Compare this result to the answer you found previously: is it possible to distinguish the two initial states from this measurement? If not, propose another measurement that would distinguish the two cases.

e) Consider a more abstract question: For the family of pure states represented by \( |\vartheta \rangle = (|+\rangle + e^{i\varphi}|-\rangle)/\sqrt{2} \), and the non-pure state \( \rho = \frac{1}{2}(|+\rangle\langle+| + |-\rangle\langle-|) \) we have \( \langle\sigma_x\rangle = 0 \). Thus a measurement of \( \sigma_x \) cannot distinguish the two states. How would you differentiate one state from another (with an appropriate measurement)? [On notations: we define the states \( |\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}, \) where \(|0\rangle, |1\rangle \) are as usual the eigenstates of the \( \sigma_z \) operator.]

**Problem 7. Creation of entanglement**

a) Consider the unitary operator \( U(\vartheta, \varphi) = \begin{pmatrix} \cos(\vartheta/2) & e^{-i\varphi} \sin(\vartheta/2) \\ -e^{i\varphi} \sin(\vartheta/2) & \cos(\vartheta/2) \end{pmatrix} \) in the usual basis \(|0\rangle, |1\rangle \). Which of the following states is entangled and which one is separable?

1. \([U(\vartheta_1, \varphi_1) \otimes U(\vartheta_1, \varphi_1)]|00\rangle\]
2. \([U(\vartheta_1, \varphi_1) \otimes U(\vartheta_2, \varphi_2)](|00\rangle + |01\rangle)/\sqrt{2}\]
3. \([U(\vartheta_1, \varphi_1) \otimes U(\vartheta_2, \varphi_2)](|00\rangle - |11\rangle)/\sqrt{2}\]

[Notice that you should be able to give an answer even without making any calculation!]

b) Consider the Hamiltonian \( \mathcal{H} = a\sigma_x\sigma_x + (1 - a)\sigma_y\sigma_y \).

1. Are its eigenstates entangled? (you can use your favorite math program to diagonalize the matrix).
2. Consider the initial state \(|00\rangle\). What is the rate of creation of entanglement by the Hamiltonian above? [Take for example as an entanglement measure the purity of the reduced state. Give an analytical expression and then you calculate the rate using your favorite program (it can also be done by hand)]]
Problem 1. Linear Algebra

a) What is an eigenvalue? Write the eigenvalue equation.

Solution:
The eigenvalues of an operator \( A \) are scalars \( \lambda \) such that \( A|\vec{v}\rangle = \lambda |\vec{v}\rangle \), where \( |\vec{v}\rangle \) is the corresponding eigenvector. This is the eigenvalue equation.

b) The operator \( C = AB - BA \) is anti-hermitian if both \( A \) and \( B \) are hermitian. Given the properties of \( C \) what is the definition of an anti-hermitian operator?

Solution:
Notice that \( C^\dagger = (AB - BA)^\dagger = BA - AB = -C \). Thus an anti-hermitian operator is defined such that \( C^\dagger = -C \).

c) The operator \( \sigma_z \) corresponds to the component of the spin of an electron along the z-axis, in units of \( \hbar/2 \).

For a state represented by the vector
\[
|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix},
\]
where \( \alpha \) and \( \beta \) are complex numbers, calculate: (i) the expectation value of \( \sigma_z \); (ii) the probability that the spin z-component is positive.

Solution:
The expectation value of \( \sigma_z \) is
\[
\langle \psi | \sigma_z | \psi \rangle = \alpha^* \beta + \beta^* \alpha = |\alpha|^2 - |\beta|^2.
\]
The probability that the spin-z component is positive is \( |\alpha|^2 \).

d) Is the state \( |\psi\rangle = \frac{1}{2}(|a\rangle|b\rangle + |b\rangle|a\rangle + |a\rangle|a\rangle + |b\rangle|b\rangle) \) entangled?

Solution:
No, since we can write it as a product state \( |\psi\rangle = \frac{1}{2}(|a\rangle + |b\rangle)(|a\rangle + |b\rangle) \).

e) Consider a flux of neutrons entering the interferometer has a random polarization. That is, each neutron has a probability \( p_\uparrow \) to have its spin pointing up and a probability \( p_\downarrow \) to be pointing down. How can you describe the state of the neutron?

Solution:
We can describe the neutron as a density operator \( \rho = p_\uparrow |\uparrow\rangle\langle\uparrow| + (1 - p_\uparrow)|\downarrow\rangle\langle\downarrow| \).

Problem 2. Trace as an inner product for the matrix vector space

a) If \( A \) and \( B \) are matrices of the same dimensions, show that \( \langle A, B \rangle = Tr[A^\dagger B] \) has all the properties of an inner product.

Solution:
To prove that it is an inner product we have to show that it satisfies the definition of inner product:
Inner product An inner product is an ordered mapping from two vectors to a complex number with the following properties:

1. positivity: \( \langle \psi | \psi \rangle \geq 0 \). The equality holds only for the zero vector \( |\psi\rangle = 0 \).
2. linearity in the second argument: \( \langle \psi \vert (c_1 \varphi_1 + c_2 \varphi_2) \rangle = c_1 \langle \psi \vert \varphi_1 \rangle + c_2 \langle \psi \vert \varphi_2 \rangle \).

3. anti-linearity in the first argument: \( \langle c_1 \varphi_1 + c_2 \varphi_2 \vert \psi \rangle = c_1^* \langle \varphi_1 \vert \psi \rangle + c_2^* \langle \varphi_2 \vert \psi \rangle \).

4. skew symmetry: \( \langle \psi \vert \varphi \rangle = \langle \varphi \vert \psi \rangle^* \) (Notice that 4+2 imply 3, so we do not really need to prove 3)

By definition, the trace maps two matrices (elements of the vector linear space of complex matrices) into a complex number. We can then show the properties

1. positivity: \( \text{Tr} \{ AA^\dagger \} = \sum_j \sum_k A_{jk}(A^\dagger)^{kj} = \sum_j \sum_k |A_{jk}|^2 \geq 0 \) with equality only if \( A_{jk} = 0, \ \forall \{j,k\} \) (the zero matrix)

2. linearity in the second argument: \( \text{Tr} \{ A^\dagger (bB) \} = \sum_j \sum_k A_{kj}^\dagger (bB)_{kj} = b \text{Tr} \{ A^\dagger B \} \) where \( b \) is a scalar (note that since trace is linear, I only show the prove wrt the scalar coefficient).

3. anti-linearity in the first argument: given a scalar \( a \), \( \text{Tr} \{ (aA)^\dagger B \} = a^* \text{Tr} \{ A^\dagger B \} \) (from the properties of the adjoint operator)

4. skew symmetry: Since \( \text{Tr} \{ A^\dagger \} = \text{Tr} \{ A^* \} = \text{Tr} \{ A \}^*, \) we have \( \langle A \vert B \rangle = \text{Tr} \{ A^\dagger B \} = \text{Tr} \{ (A^\dagger B)^\dagger \}^* = \text{Tr} \{ B^\dagger A \}^* = (B \vert A)^* \)

**Problem 3. Neutrino Oscillations**

Neutrinos can oscillates between different flavors because of the mixing of their mass eigenstate. Observing these oscillation is the goal of experiments like Opera, that measures neutrinos produced at CERN. At CERN, neutrinos are produced from kaon/pion decay, which generates muon neutrinos, e.g. \( \pi^+ \rightarrow \mu^+ + \nu_\mu \). Muon neutrinos \( \nu_\mu \) can oscillate into tau neutrinos, \( \nu_\tau \).

**a)** What is the Hamiltonian \( H \) describing the interaction that makes the neutrino oscillate between the muon and tau flavor? (Assume that the energy needed for this change is \( \Delta \) or equivalently, setting \( \hbar = 1 \), that the rate of the oscillation is \( \Delta \))

**Solution:**

This effect is characterized by the Hamiltonian

\[
H = \Delta (|\mu \rangle \langle \tau| + |\tau \rangle \langle \mu|) = \Delta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

where \( \Delta \) is a real number with the dimension of energy.

**b)** Find the normalized energy eigenkets of the Hamiltonian. What are the corresponding eigenvalues?

**Solution:**

The eigenvalues are \( \lambda_\pm = \pm \Delta \) and eigenkets \( |\pm\rangle = \frac{1}{\sqrt{3}} (|\mu \rangle \pm |\tau\rangle) \). [Notice that this simple model can be mapped onto the usual Pauli matrices describing a TLS, and the Hamiltonian is simply \( H = \Delta \sigma_x \) of which we already know the eigenkets and eigenvalues]

**c)** Suppose a neutrino is created in the state \( |\alpha\rangle = |\alpha \rangle |\mu \rangle + \beta |\tau \rangle \) at \( t = 0 \). Find the state vector at \( t > 0 \) by applying the appropriate time-evolution operator to \( |\alpha(0)\rangle \)

**Solution:**

The evolution operator is \( U(t) = e^{-iHt} \) which gives

\[
|\alpha(t)\rangle = (\cos(\Delta t) - i\beta \sin(\Delta t))|\mu\rangle + (\beta \cos(\Delta t) - i\alpha \sin(\Delta t))|\tau\rangle
\]

**d)** Suppose at \( t = 0 \) a muon neutrino is produced in a reaction. What is the probability for observing tau neutrino as a function of time?

**Solution:**

From the evolution above and setting \( \alpha = 1, \beta = 0 \), the probability \( P(t) = |\langle s \vert \alpha(t) \rangle|^2 = |\sin(\Delta t)|^2 \)
e) Suppose I suggested you that the (wrong) answer to question (a) is $H = \Delta |\tau\rangle \langle \mu|$. By explicitly solving the time-evolution problem with this Hamiltonian, show that probability conservation is violated.

Solution:
This new Hamiltonian is non Hermitian. Computing the evolution operator and the state evolution we obtain

$$|\alpha(t)\rangle = |\alpha(\mu)\rangle + [\beta - i\frac{1}{2} \alpha \Delta t] |\tau\rangle$$

The probability of being in the lower state is then $|\beta - i\frac{1}{2} \alpha \Delta t|^2$. For the same initial conditions as above we see that $P(t) = \frac{\Delta^2 t^2}{4}$ which becomes larger than 1 at long times. Also, $P(\tau) + P(\mu) = |\alpha|^2 + |\beta - i\frac{1}{2} \alpha \Delta t|^2 = 1 + \frac{1}{2}|\alpha|^2(\Delta t)^2 - 2Re[\beta \alpha] \Delta t$ which is different than 1.

f) Now assume that different neutrinos have different energies $\omega_\mu > \omega_\tau$. Write the Hamiltonian describing this energy difference in the $|\mu\rangle, |\tau\rangle$ basis.

Solution:
The Hamiltonian is described by

$$H_p = \omega_\mu |\mu\rangle \langle \mu| + \omega_\tau |\tau\rangle \langle \tau|$$

Note that I can rewrite this as

$$H_p = \frac{1}{2}(\omega_\mu + \omega_\tau) \mathbb{1} + \frac{1}{2}(\omega_\mu - \omega_\tau) \sigma_z$$

and the first term, which sets the zero of the energy, will not play a role in the evolution.

g) Calculate again the evolution of the muon neutrino (as in question (c)) when to the Hamiltonian you found in question (a) you add the Hamiltonian of question (f).

Solution:
Now the total Hamiltonian is given by $H_{tot} = H + H_p$. We first define $\delta \omega = \frac{1}{2}(\omega_\mu - \omega_\tau)$ and $\Omega = \frac{1}{2}(\omega_\mu + \omega_\tau)$. With these substitutions we find:

$$|\alpha(t)\rangle = e^{-it\Omega} \left( \frac{e^{-it(\Delta + \delta \omega) \cos[t\sqrt{\Delta^2 + \delta \omega^2}] + i(-\beta \Delta + \alpha \delta \omega) \sin[t\sqrt{\Delta^2 + \delta \omega^2}]}}{\sqrt{\Delta^2 + \delta \omega^2}} |\mu\rangle ight)$$

$$+ e^{-it\Omega} \left( \frac{e^{-it(\Delta + \delta \omega) \cos[t\sqrt{\Delta^2 + \delta \omega^2}] - i(\alpha \Delta + \beta \delta \omega) \sin[t\sqrt{\Delta^2 + \delta \omega^2}]}}{\sqrt{\Delta^2 + \delta \omega^2}} |\tau\rangle \right)$$

The probability of the neutrino to be of the tau type state if it was created in the muon type state becomes:

$$P(t) = \frac{\Delta^2}{\Delta^2 + \delta \omega^2} \sin^2 \left( t\sqrt{\Delta^2 + \delta \omega^2} \right)$$

Note that this formula is usually known as the **Rabi formula** which describes the evolution of a spin-1/2 under the action of an external magnetic field (here the potential) and of an oscillating field, seen in the rotating frame or equivalently of a two-level atom under the influence of an off-resonant laser radiation. The oscillations between the two levels are called Rabi oscillations.

The formula can be found by looking for the eigenvalues and eigenvectors of the Hamiltonian and then using the usual formula for the evolution, $\sum e^{-iE_k t/\hbar} |k\rangle$.

In this case it might have been easier to use properties of the Pauli spin matrices. We have

$$H = \Delta \sigma_z + \delta \omega \sigma_x , U = e^{-iHt}$$

Now, for any unit vector $\vec{\sigma}$, we have that $e^{-i\vec{\sigma} \cdot \vec{\sigma}} = \mathbb{1} \cos(\varphi) - i \sin(\varphi) \vec{\sigma}$. Here the vector has a length $\sqrt{\Delta^2 + \delta \omega^2}$. Thus we have:

$$U = \cos(\sqrt{\Delta^2 + \delta \omega^2} t) \mathbb{1} - i \sin(\sqrt{\Delta^2 + \delta \omega^2} t) \frac{\Delta \sigma_z + \delta \omega \sigma_x}{\sqrt{\Delta^2 + \delta \omega^2}}$$

Then, for example, if we start in $|\mu\rangle$, the only way to obtain $|\tau\rangle$ is by applying the $\sigma_z$ operator and the probability above is easily found:

$$U|\mu\rangle = [\cos(\sqrt{\Delta^2 + \delta \omega^2} t) - i \sin(\sqrt{\Delta^2 + \delta \omega^2} t) \frac{\delta \omega}{\sqrt{\Delta^2 + \delta \omega^2}}]|\mu\rangle - i \sin(\sqrt{\Delta^2 + \delta \omega^2} t) \frac{\Delta}{\sqrt{\Delta^2 + \delta \omega^2}} |\tau\rangle,$$
Problem 4. Stern-Gerlach Measurement

A beam of spin $1/2$ atoms goes through a series of Stern-Gerlach-type measurements as follows:

1. The first measurement accepts $s_z = \hbar/2$ atoms and rejects $s_z = -\hbar/2$ atoms (where $s_z$ is the eigenvalue of the spin angular momentum $S_z$).

2. The second measurement accepts $s_n = \hbar/2$ atoms and rejects $s_n = -\hbar/2$ atoms, where $s_n$ is the eigenvalue of the operator $S \cdot \hat{n}$, with $\hat{n}$ making an angle $\beta$ in the $xz$-plane with respect to the $z$-axis.

3. The third measurement accepts $s_z = -\hbar/2$ atoms and rejects $s_z = \hbar/2$ atoms.

a) What is the intensity of the final beam when the $s_z = \hbar/2$ beam surviving the first measurement is normalized to unity?

b) How must we orient the second measuring apparatus to maximize the intensity of the final beam?

Solution:

We choose a basis $|\uparrow\rangle$, $|\downarrow\rangle$, corresponding to $s_z = \pm \hbar/2$. The first measurement is represented by the projector $|\uparrow\rangle \langle \uparrow|$. Given any initial state $|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$ it projects it to $a|\uparrow\rangle$. Since we assume to renormalize this state, it is equivalent to say that the first measurement prepares the beam in the state $|\psi\rangle_1 = |\uparrow\rangle$.

The second measurement is the projector

\[
(\cos(\beta/2)|\uparrow\rangle + \sin(\beta/2)|\downarrow\rangle)(\cos(\beta/2)|\uparrow\rangle + \sin(\beta/2)|\downarrow\rangle)
\]

\[
= \cos^2(\beta/2)|\uparrow\rangle\langle \uparrow| + \cos(\beta/2)\sin(\beta/2)|\uparrow\rangle\langle \downarrow| + \sin(\beta/2)|\downarrow\rangle\langle \uparrow| + \sin^2(\beta/2)|\downarrow\rangle\langle \downarrow|
\]

The probability of “surviving” this second measurement is $\cos^2(\beta/2)$ and the (normalized) state is $|\psi\rangle_2 = \cos(\beta/2)|\uparrow\rangle + \sin(\beta/2)|\downarrow\rangle$. Here I’m assuming that the measurements in the Stern-Gerlach apparatus act as filters, if the state does not correspond to the given eigenvalue, the spin is “absorbed” (this is similar to the polarizer example we saw in class). Considering both measurement results gives a slightly different answer. Finally, the probability of passing the third measurement is $\sin^2(\beta/2)$, so that the intensity of the final beam (as given by the probability of observing a beam at the end of the measurements) is $I(\beta) = \sin^2(\beta/2)\cos^2(\beta/2)$.

In order to maximize the final beam intensity, we want to maximize $I(\beta)$, so we choose $\beta = \pi/2$; the optimal intensity is $1/4$ of the initial intensity.

Problem 5. Neutron Interferometry

Consider a neutron interferometer (NI), such as the Mach-Zehnder interferometer in the figure. We send in a beam of neutrons. We assume that the flux of neutrons is so low (neutrons can be very slow) so that only one neutron is present at any time inside the interferometer. The first beamsplitter divides the neutron flux into two parts, that will go into the upper arm or the lower arm. The lower and upper beams are then reflected at the mirrors and recombined at the second beam splitter, after which the neutron flux is measured at one arm. We assume that both beamsplitter works in the same way, delivering an equal flux to each arm (that is, the transmission and reflection are the same).

a) Define the (minimal) Hilbert space describing this problem (e.g. give the basis spanning the space)
Solution:

Although the neutron can be characterized by many different parameters (e.g. their position \( \vec{r} \), momentum \( \vec{p} \), spin, etc.) to describe the neutron interferometer behavior for the purpose of this problem we only need to describe the neutron as a two-level system. For example, we can describing the neutron as being in the upper or lower arm. A more precise description would be the following: define \( x \) to be the transversal direction in the interferometer (the vertical direction in the drawing). Then we define two states, for positive or negative momentum \( \pm p_z \) of the neutron. The Hilbert space describing the system is a 2 \( \times \) 2 Hilbert space, spanned by the basis \( |U\rangle, |L\rangle \) (or \( |+p_z\rangle = |U\rangle \) and \( |-p_z\rangle = |L\rangle \).

b) **What is the propagator describing the action of the Beamsplitter?**

Solution:

From the problem statement we now: \( U_{BS}|U\rangle = (|U\rangle + |L\rangle)/\sqrt{2} \) and we also have the condition that \( U_{BS} \) is a unitary operator, so that \( U_{BS}(|U\rangle + |L\rangle)/\sqrt{2} = |U\rangle \). From the first condition we find:

\[
U|U\rangle = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

so \( u_{11} = u_{21} = \frac{1}{\sqrt{2}} \). From the second condition (unitarity) we have \( u_{12}^* = -u_{22} \) and \(|u_{12}|^2 + |u_{22}|^2 = 1 \) (since e.g. \( U|L\rangle = u_{12}|U\rangle + u_{22}|L\rangle \) must be a normalized vector). We thus have two solutions: \( u_{12} = -\frac{i}{\sqrt{2}}, u_{22} = \frac{1}{\sqrt{2}} \) or \( u_{12} = -\frac{1}{\sqrt{2}}, u_{22} = \frac{1}{\sqrt{2}} \). In the first case:

\[
U_{BS} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}
\]

is the Hadamard matrix, while in the second case \( U_{BS} = e^{-i\sigma_z \pi/4} \) is a rotation around \( y \). Both choices give the same result in the end.

As for the description of the mirror, you can either describe it as a perfect reflection \( U_m = \sigma_x \) (this is more appropriate for the description in terms of momenta) or as a perfect transmission \( U_m = 1 \). As for our purposes both descriptions give the same results, I’ll adopt the second, simpler one). Also, notice that a ”lossy” mirror (such not all neutrons are reflected back into the interferometer, but leak out of the system) cannot be fully described by just our TLS Hilbert space.

Finally, \( U_{BS} \) can be more generally described in terms of transmission and reflection coefficients

\[
U_{BS} = \begin{pmatrix} \sqrt{t} & \sqrt{r} \\ \sqrt{r} & -\sqrt{t} \end{pmatrix}
\]

such that \( r + t = 1 \) for conservation of probabilities. Since transmission and reflection are positive quantities, usually other possible choices \( U_{BS} = e^{i\pi \sigma_z/4} \) or \( (\sigma_z \pm \sigma_y)\sqrt{2} \) are not used.

c) **What is the state at the position 1, assuming the neutron was initially traveling upward before entering the NI?**

Solution:

\[ |\psi\rangle_1 = (|U\rangle + |L\rangle)/\sqrt{2}. \]

d) **What is the probability of observing a neutron in the upper arm detector?**

Solution:

\[ |\psi\rangle_2 = |U\rangle, \] thus, if our observable is the number of neutron in the upper arm, the measurement always returns 1 with certainty (probability =1 ).

e) **We now introduce an object in the lower path. This modifies the momentum of the neutron, and its effect is seen as an added phase to the neutrons passing through the lower path. Write the operator describing this phase shift and calculate again the probability of measuring a neutron at the upper path detector.**

Solution:

The object behaves as a phase flag, such that \((|U\rangle + |L\rangle)/\sqrt{2} \rightarrow (|U\rangle + e^{i\varphi}|L\rangle)/\sqrt{2} \). The flag operator can thus be written as \( U_f = e^{i\varphi/2}e^{-i\varphi \sigma_z/2} \):

\[
U_f = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}
\]
Now after the second BS, the state of the neutron is $|\psi\rangle_2 = e^{i\varphi/2}(|\psi\rangle(U) - i \sin(\varphi/2)|L\rangle)$ so that the probability of measuring a neutron at the upper path detector is $\cos(\varphi/2)^2$. (Notice this result is for $U_{BS} = \text{Had}$, while for $U_{BS} = e^{-i\pi\sigma_y/4}$ we obtain $\sin(\varphi/2)^2$.

f) The usual signal for interferometers is the contrast $C = |(S_U - S_L)/(S_U + S_L)|$, where $S_U(S_L)$ is the signal (# of neutrons) at the upper(lower) detector. Indeed, this is always necessary since we need to calibrate and normalized the signal. What is the contrast for the neutron interferometer if the added phase (see previous question) is $\varphi = \pi/2$?

Bonus: what is the operator describing the observable measured by the contrast?

Solution:
The observable in the upper path is $O_u = |U\rangle\langle U|$, so that $S_u = \langle O_U \rangle$ (and similarly for the lower path). Thus the observable for the contrast is

$$\hat{C} = \frac{|U\rangle\langle U| - |L\rangle\langle L|}{|U\rangle\langle U| + |L\rangle\langle L|} = \frac{\sigma_x}{\mathbb{1}} = \sigma_z$$

The state for $\varphi = \pi/2$ is $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|U\rangle - i|L\rangle)$ which is an eigenstate of $\sigma_y$ and thus has $C = \langle \sigma_z \rangle = 0$.

Problem 6. Pure vs. Mixed States

Consider again the NI of Problem 5. We previously assumed that the beam of neutrons all had the same (upward) momentum $|\psi(0)\rangle = |U\rangle$. However, if the neutrons arrive from a reactor, we might not be able to control their initial state.

b) Assume the neutrons can be represented by a mixed-state, with 50-50 probability of having initially an upward $|U\rangle$ or downward $|L\rangle$ momentum. What is the operator describing this state?

Solution:
It is the completely mixed state $\rho = \frac{1}{2}|U\rangle\langle U| + \frac{1}{2}|L\rangle\langle L| = \frac{1}{2}\mathbb{1}$. c) For the initial state described above, what is the measured contrast for the same condition as in Problem 5? Compare this result to the answer you found previously: is it possible to distinguish the two initial states from this measurement? If not, propose another measurement that would distinguish the two cases.

Solution:
We have that $C = \text{Tr}\{\sigma_z \rho\} = \text{Tr}\{\sigma_z \mathbb{1}/2\} = 0$, thus we obtain the same result as before. To distinguish experimentally the two states, we could measure the state after letting it through another different beamsplitter, with

$$U'_{BS} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

In this case, the measurement for the mixed state still gives zero, while it gives a non-zero result for the first, pure state. A second strategy is to change the phase $\varphi$: the mixed state will always give a zero result, while the pure state will have a varying contrast.

e) Consider a more abstract question: For the family of pure states represented by $|\vartheta\rangle = (|+\rangle + e^{i\vartheta}|-\rangle)/\sqrt{2}$, and the non-pure state $\rho = \frac{1}{2}(|+\rangle\langle +| + |-\rangle\langle -|)$ we have $\langle \sigma_z \rangle = 0$. Thus a measurement of $\sigma_x$ cannot distinguish the two states. How would you differentiate one state from another (with an appropriate measurement)? [On notations: we define the states $|\pm\rangle = (|0\rangle \pm |1\rangle)/\sqrt{2}$, where $|0\rangle, |1\rangle$ are as usual the eigenstates of the $\sigma_z$ operator.]

Solution:
Notice that $|+\rangle$ is the eigenstate of $\sigma_x$ with eigenvalue 1 and $|-\rangle$ with eigenvalue $-1$. Thus

$$\langle \vartheta | \sigma_x | \vartheta \rangle = \langle + | \sigma_x | + \rangle + \langle - | \sigma_x | - \rangle = 0.$$ 

Also notice that $\rho = \frac{1}{2}(|+\rangle\langle +| + |-\rangle\langle -|)$ is just the identity $\mathbb{1}$ written in the $\sigma_x$-basis. Since $\langle \sigma_x \rangle = \text{Tr}\{\sigma_x \rho\}$ is independent of the basis, we have $\langle \sigma_x \rangle = \text{Tr}\{\sigma_x\} = 0$. 

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When we measure the expectation value of \(\sigma_z\) these two states (pure, coherent superposition, the first one, and mixed, incoherent superposition the second) are indistinguishable. If instead, for example, we measure \(\langle \sigma_z \rangle\) we obtain two different results. \(\langle \sigma_z \rangle = \text{Tr} \{\rho \sigma_z \} = 0\) for the same reasons as before (\(\rho = \mathbb{1}\)), while \(\langle \sigma_z \rangle = \langle \vartheta \sigma_z | \vartheta \rangle = 2 \cos \vartheta\). We can in fact write \(|\vartheta \rangle = e^{i \vartheta/2}(\cos(\vartheta/2)|0\rangle - i \sin(\vartheta/2)|1\rangle) \sqrt{2}\) from which the result above follows. Except for \(\vartheta = \pi/2\) then the two results are different.

- More generally, we will need to measure two (or more) non-compatible (non-commuting) observable to distinguish between a pure-state coherent superposition and a mixed-state incoherent superposition. (A full reconstruction of the state, and calculating the purity of it, is another way, but often it is not necessary).

**Problem 7. Creation of entanglement**

a) Consider the unitary operator \(U(\vartheta, \varphi) = \begin{pmatrix} \cos(\vartheta/2) & e^{-i \varphi} \sin(\vartheta/2) \\ -e^{i \varphi} \sin(\vartheta/2) & \cos(\vartheta/2) \end{pmatrix}\) in the usual basis \(|0\rangle, |1\rangle\).

Which of the following states is entangled and which one is separable?

1. \([U(\vartheta_1, \varphi_1) \otimes U(\vartheta_1, \varphi_1)]|00\rangle\)
2. \([U(\vartheta_1, \varphi_1) \otimes U(\vartheta_2, \varphi_2)](|00\rangle + |01\rangle)/\sqrt{2}\)
3. \([U(\vartheta_1, \varphi_1) \otimes U(\vartheta_2, \varphi_2)](|00\rangle - |11\rangle)/\sqrt{2}\)

[Notice that you should be able to give an answer even without making any calculation!]

**Solution:**

Since the propagator acts on each individual subsystem it cannot change the entanglement of the system’s state. As the initial state is separable in the first two cases and entangled for the other state this is true for the evolved states too.

b) Consider the Hamiltonian \(H = a \sigma_x \sigma_x + (1-a) \sigma_y \sigma_y\).

1. Are its eigenstates entangled? (you can use your favorite math program to diagonalize the matrix).

2. Consider the initial state \(|00\rangle\). What is the rate of creation of entanglement by the Hamiltonian above?

[Take for example as an entanglement measure the purity of the reduced state. Give an analytical expression and then you calculate the rate using your favorite program (it can also be done by hand)!]

**Solution:**

The eigenstates are

\[|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle), \quad E = 2a - 1\]

\[|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle + |1\rangle|0\rangle), \quad E = 1\]

\[|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle|1\rangle - |1\rangle|0\rangle), \quad E = -1\]

\[|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle - |1\rangle|1\rangle), \quad E = -2a + 1\]

Thus, the eigenstates are the four Bell states and they are entangled. For \(a = 1, 0\) the eigenvalues are degenerate, so it is possible to choose non-entangled eigenstates: \(a = 1, |++\rangle, |+-\rangle, |-+\rangle, |--\rangle\), where \(|\pm\rangle\) are eigenstates of \(\sigma_x\) and similarly, tensor products of eigenstates of \(\sigma_y\) for \(a = 0\).

Notice that \(|00\rangle = \frac{1}{\sqrt{2}}(|\Phi^+\rangle + |\Phi^-\rangle)\), thus the evolution of the state \(|00\rangle\) under the given Hamiltonian is

\[\frac{1}{\sqrt{2}}(e^{i(1-2a)t}|\Phi^+\rangle + e^{-i(1-2a)t}|\Phi^-\rangle) = \cos(t(1 - 2a))|00\rangle + i \sin(t(1 - 2a))|11\rangle\]

[Notice that for \(a = \frac{1}{2}\) there’s no evolution]
We could have also calculated $U = e^{-i(\sigma_x \sigma_x + (1-a)\sigma_y \sigma_y)t}$. Notice that since $[\sigma_z \sigma_z, \sigma_y \sigma_y] = 0$ we can simplify this exponential as

$$U = e^{-ia\sigma_x \sigma_x} e^{i((1-a)\sigma_y \sigma_y)t} = [\cos(at) \mathbb{1} - i \sin(at) \sigma_x \sigma_x][\cos((1-a)t) \mathbb{1} - i \sin((1-a)t) \sigma_y \sigma_y]$$

$$= \cos(at) \cos((1-a)t) \mathbb{1} - \sin(at) \sin((1-a)t) \sigma_z \sigma_z - i(\cos(at) \sin((1-a)t) \sigma_y \sigma_y + \sin(at) \cos((1-a)t) \sigma_x \sigma_x).$$

Applied to $|00\rangle$ and using trigonometric sum relationships, we obtain the same result as before.

The reduced state is then $\rho_1 = \frac{1}{2}(\mathbb{1} \cos(2t(1-2a)) + \sigma_z)$. The purity is then $P(t) = \frac{1}{4}(\cos(4(1-2a)t) + 3)$ and the rate at which purity changes is

$$\dot{P} = (2a - 1) \sin(4(1-2a)t)$$