1.1 Consider the discrete-time system specified by the input/output relationship

\[ y[n] = x[-(n^2)]. \quad (1) \]

Determine whether or not this system possesses each of the following properties: linearity, time-invariance, causality, stability.

For linearity, suppose we have inputs \( x_1[n] \) and \( x_2[n] \) with corresponding outputs \( y_1[n] \) and \( y_2[n] \). Let \( x_3[n] = a_1 x_1[n] + a_2 x_2[n] \). If we pass this signal through our system, we get

\[ y_3[n] = x_3[-(n^2)] = a_1 x_1[-(n^2)] + a_2 x_2[-(n^2)] = a_1 y_1[n] + a_2 y_2[n]. \]

Since \( y_3[n] = a_1 y_1[n] + a_2 y_2[n] \), we see that this system is linear.

For time-invariance, suppose we have input \( x[n] \) with corresponding output \( y[n] = x[-(n^2)] \).

Now we consider input \( x'[n] = x[n - n_0] \) with corresponding output \( y'[n] = x'[-(n^2)] = x[-(n^2) - n_0] \). However, in general,

\[ y[n - n_0] = x[-(n - n_0)^2] = x[-(n^2) + 2nn_0 - n_0^2] \neq x[-(n^2) - n_0] = y'[n]. \]

Therefore, the system is not time-invariant.

For causality, we note that for any integer \( n \),

\[ |n| \leq |n^2| \implies -(n^2) \leq n. \]

Therefore, since \( y[n] = x[-(n^2)] \), we know that \( y[n] \) never relies on future values of our input \( x[n] \). Therefore, our system is causal.

For stability, we note that every output value is a value from our input signal. Therefore, if our input is bounded, our output is bounded, as well. Thus, the system is stable.

1.2 When the input to an LTI system is

\[ x[n] = 5u[n] \quad (2) \]

the output is

\[ y[n] = \left[ 2 \left( \frac{1}{2} \right)^n + 3 \left( -\frac{3}{4} \right)^n \right] u[n]. \quad (3) \]

(a) Determine the system function \( H(z) \) for this system. Sketch its pole-zero diagram and indicate the region of convergence of \( H(z) \).
We know that $H(z) = \frac{Y(z)}{X(z)}$, where $Y(z)$ and $X(z)$ are the $z$-transforms of $y[n]$ and $x[n]$, respectively. We have

$$Y(z) = \sum_{k=-\infty}^{\infty} y[k] z^{-k} = \sum_{k=0}^{\infty} \left(2 \left(\frac{1}{2}\right)^k + 3 \left(-\frac{3}{4}\right)^k\right) z^{-k}$$

$$= 2 \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k + 3 \sum_{k=0}^{\infty} \left(-\frac{3}{4}\right)^k = \frac{2}{1 - \frac{1}{2} z^{-1}} + \frac{3}{1 + \frac{3}{4} z^{-1}} = \frac{5}{(1 - \frac{1}{2} z^{-1})(1 + \frac{3}{4} z^{-1})},$$

requiring that $|z| > 3/4$. Similarly, we have

$$X(z) = \sum_{k=-\infty}^{\infty} x[k] z^{-k} = \sum_{k=0}^{\infty} 5 z^{-k} = \frac{5}{1 - z^{-1}},$$

requiring that $|z| > 1$. Therefore, we find

$$H(z) = \frac{Y(z)}{X(z)} = \frac{5(1 - z^{-1})}{5(1 - \frac{1}{2} z^{-1})(1 + \frac{3}{4} z^{-1})} = \frac{1 - z^{-1}}{(1 - \frac{1}{2} z^{-1})(1 + \frac{3}{4} z^{-1})}.$$

We see that the poles of this system are $\frac{1}{2}$ and $-\frac{3}{4}$, and the zeros are 0 and 1 (the zero at 0 can be seen by the $z^{-2}$ term in the denominator relative to the $z^{-1}$ in the numerator). Thus, we get the following pole-zero plot:

![Pole-Zero Plot](image)

Since $x[n]$ is right-sided, we know that the ROC of $X(z)$ must be outward facing. Given its single pole at $z = 1$, the ROC of $X(z)$ is $|z| > 1$. In order to have any overlap in ROC with $X(z)$, we therefore must choose the ROC of $H(z)$ to be $|z| > \frac{3}{4}$.

(b) Determine the impulse response of the system, $h[n]$. 
By partial fraction decomposition, we get

\[ H(z) = \frac{1 - z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{3}{4}z^{-1})} = \frac{-2/5}{1 - \frac{1}{2}z^{-1}} + \frac{7/5}{1 + \frac{3}{4}z^{-1}}. \]

Since the ROC of \( H(z) \) is outward facing, we know that \( h[n] \) is right-sided. Given this, \((1 - \frac{1}{2}z^{-1})^{-1}\) corresponds to \((\frac{1}{2})^nu[n]\), and \((1 + \frac{3}{4}z^{-1})^{-1}\) corresponds to \((-\frac{3}{4})^nu[n]\). Therefore, we get

\[ h[n] = \left( -\frac{2}{5} \left( \frac{1}{2} \right)^n + \frac{7}{5} \left( -\frac{3}{4} \right)^n \right) u[n]. \]

(c) Write the difference equation that characterizes the system.

We have

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2}} \]

\[ \implies Y(z) \left( 1 + \frac{1}{4}z^{-1} - \frac{3}{8}z^{-2} \right) = X(z) \left( 1 - z^{-1} \right). \]

Since a factor of \( z \) corresponds to a single time step shift, we get

\[ y[n] + \frac{1}{4}y[n-1] - \frac{3}{8}y[n-2] = x[n] - x[n-1]. \]